Math 4550 - Homework # 1 - Groups

Part 1 - Computations

1. (a) Calculate the group table for $\mathbb{Z}_2$, and (b) find the inverse of each element in $\mathbb{Z}_2$.

2. (a) Calculate the group table for $\mathbb{Z}_4$, and (b) find the inverse of each element in $\mathbb{Z}_4$.

3. (a) List the elements of $\mathbb{Z}_6$, and (b) find the inverse of each element of $\mathbb{Z}_6$.

4. (a) Draw a picture of U_4, (b) calculate the group table for U_4, and (c) find the inverse of each element in U_4.

5. (a) Draw a picture of U_6, and (b) find the inverse of each element in U_6.

6. (a) List the elements of D_6, (b) calculate the group table for D_6, and (c) find the inverse of each element in D_6.

7. (a) List the elements of D_8, (b) calculate rsr^3 and $(sr^3)(sr^2)$ and $r^2r^3srsr^{-2}$, and (c) find the inverses of the elements r, r^2, sr, and sr^2.

8. For each A and B, verify that A and B are in $GL(2,\mathbb{R})$. Then calculate A^{-1}, B^{-1} and AB.

   (a) $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1/2 & 5 \end{pmatrix}$ (b) $A = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$

Part 2 - Proofs

Note: For abstract groups I'm using $a * b$ in the problems below. After this HW we will drop this notation and just write ab like we are doing in class

9. In D_{2n} prove that $(sr^k)^{-1} = sr^k$.

10. Let G be a group and let $a, b, c \in G$. Prove that if $a * b = a * c$, then $b = c$.

11. Let G be a group where every element is it's own inverse. Prove that G is abelian.

12. Consider $\mathbb{Z}_{10} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}\}$.

    (a) Show that $\mathbb{Z}_{10}$ is not a group under multiplication.

(b) Consider the subset $G = \{\overline{2}, \overline{4}, \overline{6}, \overline{8}\}$ of $\mathbb{Z}_{10}$. Show that G is a group under multiplication. State what the identity element is and the inverses of each element.

13. Show that $\mathbb{Z} - \{0\}$ is not a group under multiplication.

14. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Show that $\mathbb{R}^+$ is not a group under the operation $a * b = \sqrt{ab}$.

15. Prove that $\mathbb{R} - \{-1\}$ is a group under the operation $a * b = a + b + ab$.

    For example, $5 * (-2) = 5 + (-2) + (5)(-2) = -7$.

16. Let G be an abelian group and $a, b \in G$. Prove by induction that $(a * b)^n = (a^n) * (b^n)$ for all positive integers n.
